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Exact values for the cubic lattice Green functions

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Abstract. It is shown that the value of the cubic lattice Green functions $G(z; l, m, n)$ at the upper critical point $z = 3$ (for the simple cubic and face-centred cubic lattices) or $z = 1$ (for the body-centred cubic lattice) is expressible rationally in terms of the known value of $G(3; 0, 0, 0)$ or $G(1; 0, 0, 0)$, and π.

1. Introduction

The simple cubic lattice Green function is given by

$$G(z; l, m, n) = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{\cos(lu)\cos(mv)\cos(nw)}{z - \cos u - \cos v - \cos w} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}w \qquad (1)$$

for $z \geqslant 3$, where, throughout this paper, l, m, n denote integers. For convenience, we shall use the notations $g_l(z) = G(z; l, 0, 0)$, $g_l = g_l(3)$. In his classic paper Watson [1] proved that

$$\begin{aligned} g_0 &= \frac{2\sqrt{2}}{\pi^2} k'_+ k'_- K_+ K_- \\ &= \frac{4}{\pi^2}(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6})K_-^2 = 0.505\,462\,0197 \end{aligned} \qquad (2)$$

where $K_\pm = \boldsymbol{K}(k_\pm)$ is the complete elliptic integral of the first kind with modulus

$$k_\pm = (2 - \sqrt{3})(\sqrt{3} \pm \sqrt{2}).$$

It was subsequently shown by Glasser and Zucker [2] that

$$g_0 = \frac{\sqrt{6}}{96\pi^3} \Gamma\left(\frac{1}{24}\right) \Gamma\left(\frac{5}{24}\right) \Gamma\left(\frac{7}{24}\right) \Gamma\left(\frac{11}{24}\right) \qquad (3)$$

which, by means of an identity found by Borwein and Zucker [3], can be reduced to

$$g_0 = \frac{\sqrt{3} - 1}{96\pi^3} \Gamma^2\left(\frac{1}{24}\right) \Gamma^2\left(\frac{11}{24}\right). \qquad (4)$$

The function $g_0(z)$ is of considerable importance in random walk theory, statistical mechanics, solid state theory, and a number of other areas, and has been studied extensively. Although it can be expressed as the Kampé de Fériet function [4]

$$g_0(z) = \frac{1}{z} F^{1:0;2}_{2:1;0} \left[\begin{matrix} \frac{1}{2}: & -; & \frac{1}{2}, & 1; & \frac{4}{z^2}, \frac{1}{z^2} \\ 1, & 1: & 1; & -; & \end{matrix} \right]$$

it was not evaluated in practical form until 1973 when in an important paper Joyce [5] proved that

$$g_0(z) = \frac{4}{\pi^2 z} \frac{(1 - 3x_1/4)^{1/2}}{1 - x_1} \boldsymbol{K}(k_+(z))\boldsymbol{K}(k_-(z)) \tag{5}$$

where

$$k_\pm^2(z) = \tfrac{1}{2} \pm \tfrac{1}{4} x_2 (4 - x_2)^{1/2} - \tfrac{1}{4}(2 - x_2)(1 - x_2)^{1/2}$$

$$x_1 = \frac{z^2 + 3 - (z^2 - 9)^{1/2}(z^2 - 1)^{1/2}}{2z^2} \qquad x_2 = \frac{x_1}{x_1 - 1}.$$

Basically, this was achieved by expanding $g_0(z)$ as a power series in z^{-1}, finding the recursion relation obeyed by the coefficients, and identifying this as the recursion relation associated with the series solution of a certain Fuchsian differential equation of third order. A theorem of Appell shows that this solution is expressible as the square of a Heun function, which Joyce next reduced to the product of two complete elliptic integrals $\boldsymbol{K}$. In a later paper Joyce [6] showed that $g_0(z)$ is expressible as the square of an elliptic integral $\boldsymbol{K}$ with real modulus, namely,

$$g_0(z) = \frac{4}{\pi^2 z} \frac{(1 - 3x_1/4)^{1/2}}{1 - x_1} \frac{(4 - x_2)^{1/2} - (1 - x_2)^{1/2}}{1 - k_-^2(z)} [\boldsymbol{K}(k(z))]^2 \tag{6}$$

where

$$k^2(z) = \frac{-k_-^2(z)}{1 - k_-^2(z)}.$$

For $z = 3$ the expression (6) reduces to Watson's result in (2).

Our aim is to determine exact values, in terms of the gamma function, for $G(3; l, m, n)$ at lattice points $lmn \neq 000$. A procedure for doing this is based on the recursion scheme due to Duffin and Shelly [7], who showed that $G(3; l, m, n)$ is expressible rationally in terms of g_0 and $G(3; 1, 1, 0)$. In the next section we work out explicit expressions for the Green functions of the simple cubic, face-centred cubic, and body-centred cubic lattices. In the concluding section we apply these results to evaluate several Bessel function integrals, and we present some new results for the effective resistance in a cubic lattice network of unit resistors.

2. Calculations

The recursion scheme developed by Duffin and Shelly [7, pp 224–5], leads to the representation

$$G(3; l, m, n) = \lambda_1 g_0 + \lambda_2 G(3; 1, 1, 0) + \lambda_3 (g_1 - g_0) \tag{7}$$

where λ_1, λ_2, λ_3 are rational numbers. Numerical values of λ_1, λ_2, λ_3 are presented in [7, table 2], for lmn ranging from 000–555, subject to $l \geqslant m \geqslant n$. To determine g_1 and $G(3; 1, 1, 0)$, we employ the property that $G(z; l, m, n)$ is a solution of the partial difference equation

$$zG(z; l, m, n) - \tfrac{1}{2}[G(z; l+1, m, n) + G(z; l-1, m, n) + G(z; l, m+1, n) + G(z; l, m-1, n) + G(z; l, m, n+1) + G(z; l, m, n-1)] = \delta_{l0}\delta_{m0}\delta_{n0}$$

where δ_{l0} denotes the Kronecker delta. By setting $m = n = 0$, and using the symmetry of $G(z; l, m, n)$ in l, m, n, we obtain

$$G(z; l, 1, 0) = \tfrac{1}{2} z g_l(z) - \tfrac{1}{4} g_{l+1}(z) - \tfrac{1}{4} g_{l-1}(z) - \tfrac{1}{2}\delta_{l0}. \tag{8}$$

Next we set $z = 3$ and $l = 0, l = 1$, in (8) to find

$$g_1 = g_0 - \tfrac{1}{3} \tag{9}$$

$$G(3; 1, 1, 0) = \tfrac{3}{2}g_1 - \tfrac{1}{4}g_2 - \tfrac{1}{4}g_0 = \tfrac{5}{4}g_0 - \tfrac{1}{4}g_2 - \tfrac{1}{2}. \tag{10}$$

Substitution of (9) and (10) into (7) shows that $G(3; l, m, n)$ is rationally expressible in terms of g_0 and g_2.

To determine g_2, we employ the recurrence formula

$$g_2(z) = -\tfrac{2}{3}z + \tfrac{1}{3}(2z^2 - 1)g_0(z) + \tfrac{4}{3}z(z^2 - 5)g_0'(z) + \tfrac{2}{3}(z^2 - 1)(z^2 - 9)g_0''(z) \tag{11}$$

due to Horiguchi and Morita [8]. The evaluation of (11) at $z = 3$ requires some care, since $g_0'(z)$ and $g_0''(z)$ become singular at $z = 3$. Accordingly, we expand $g_0(z)$ around $z = 3$, and then take the derivatives. The expansion needed is given in [5, form (5.24)], from which we obtain

$$g_0(z) = g_0 + \alpha(z - 3)^{1/2} + \beta(z - 3) + \mathrm{O}((z - 3)^{3/2}) \tag{12}$$

where

$$\alpha = -\frac{1}{\pi\sqrt{2}} \qquad \beta = -\frac{7}{48}g_0 + \frac{1}{8\pi^2 g_0}.$$

Differentiation of (12) yields

$$g_0'(z) = \frac{\alpha}{2}(z - 3)^{-1/2} + \beta + \mathrm{O}((z - 3)^{1/2})$$

$$g_0''(z) = -\frac{\alpha}{4}(z - 3)^{-3/2} + \mathrm{O}((z - 3)^{-1/2}).$$

Since $g_0'(z)$ and $g_0''(z)$ will be multiplied by 16 and 32 $(z - 3)$, respectively, the singular terms in the expansion of $g_2(z)$ cancel. Furthermore, none of the higher-order terms contribute at $z = 3$, so that

$$g_2 = -2 + \frac{17}{3}g_0 + 16\beta = \frac{10}{3}g_0 + \frac{2}{\pi^2 g_0} - 2. \tag{13}$$

From the previous results it is clear that $G(3; l, m, n)$ is expressible in the form

$$G(3; l, m, n) = r_1 g_0 + \frac{r_2}{\pi^2 g_0} + r_3 \tag{14}$$

where r_1, r_2, r_3 are rational numbers. These numbers are related to Duffin and Shelly's parameters $\lambda_1, \lambda_2, \lambda_3$ from (7) by

$$r_1 = \lambda_1 + \tfrac{5}{12}\lambda_2 \qquad r_2 = -\tfrac{1}{2}\lambda_2 \qquad r_3 = -\tfrac{1}{3}\lambda_3. \tag{15}$$

Various values of r_1, r_2, r_3, obtained from [7, table 2] and (15), are shown in table 1.

Next we turn to the face-centred cubic lattice Green function given by

$$F(z; l, m, n) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos(lu)\cos(mv)\cos(nw)}{z - \cos u\cos v - \cos v\cos w - \cos u\cos w}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \tag{16}$$

for $z \geqslant 3$, and $l + m + n$ even ($F = 0$ when $l + m + n$ is odd). Watson [1] showed that

$$f_0 = F(3; 0, 0, 0) = \frac{\sqrt{3}}{\pi^2}[\boldsymbol{K}(k)]^2 = \frac{3\Gamma^6(\frac{1}{3})}{2^{14/3}\pi^4} = 0.448\,220\,3944 \tag{17}$$

where

$$k = \sin\frac{\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

Table 1.

$G(3; l, m, n) = r_1 g_0 + r_2/(\pi^2 g_0) + r_3$							
lmn	r_1	r_2	r_3	lmn	r_1	r_2	r_3
000	1	0	0	330	$\frac{50}{3}$	$-\frac{1046}{25}$	0
100	1	0	$-\frac{1}{3}$	430	$\frac{2641}{48}$	$-\frac{28\,049}{200}$	$\frac{1}{3}$
200	$\frac{10}{3}$	2	-2	530	$\frac{3589}{18}$	$-\frac{1993\,883}{3675}$	8
300	$\frac{35}{2}$	21	-13	331	$-\frac{35}{3}$	$\frac{148}{5}$	0
400	$\frac{994}{9}$	$\frac{542}{3}$	-92	431	$-\frac{1505}{36}$	$\frac{110\,851}{1050}$	0
500	$\frac{9287}{12}$	$\frac{3005}{2}$	$-\frac{2077}{3}$	531	$-\frac{1329}{8}$	$\frac{297\,981}{700}$	$-\frac{4}{3}$
110	$\frac{5}{12}$	$-\frac{1}{2}$	0	332	$\frac{35}{9}$	$-\frac{1012}{105}$	0
210	$\frac{3}{8}$	$-\frac{9}{4}$	$\frac{1}{3}$	432	$\frac{525}{32}$	$-\frac{4617}{112}$	0
310	$-\frac{79}{36}$	$-\frac{85}{6}$	4	532	$\frac{2555}{36}$	$-\frac{187\,777}{1050}$	0
410	$-\frac{515}{16}$	$-\frac{879}{8}$	$\frac{115}{3}$	333	$-\frac{35}{16}$	$\frac{1587}{280}$	0
510	$-\frac{11\,617}{36}$	$-\frac{138\,331}{150}$	348	433	$-\frac{595}{72}$	$\frac{8809}{420}$	0
111	$-\frac{1}{8}$	$\frac{3}{4}$	0	533	$-\frac{2233}{48}$	$\frac{164\,399}{1400}$	0
211	$-\frac{2}{3}$	2	0	440	$\frac{6025}{36}$	$-\frac{620\,161}{1470}$	0
311	$-\frac{11}{4}$	$\frac{21}{2}$	$-\frac{2}{3}$	540	$\frac{18\,471}{32}$	$-\frac{28\,493\,109}{19\,600}$	$-\frac{1}{3}$
411	$-\frac{9}{2}$	$\frac{357}{5}$	-12	441	$-\frac{4165}{32}$	$\frac{919\,353}{2800}$	0
511	$\frac{275}{4}$	$\frac{5751}{10}$	-150	541	$-\frac{1390}{3}$	$\frac{286\,274}{245}$	0
220	$\frac{73}{36}$	$-\frac{29}{6}$	0	442	$\frac{2975}{48}$	$-\frac{31\,231}{200}$	0
320	$\frac{319}{48}$	$-\frac{119}{8}$	$-\frac{1}{3}$	542	$\frac{7777}{32}$	$-\frac{1715\,589}{2800}$	0
420	$\frac{2183}{72}$	$-\frac{13\,903}{300}$	-6	443	$-\frac{539}{32}$	$\frac{119\,271}{2800}$	0
520	$\frac{2897}{16}$	$-\frac{15\,123}{200}$	$-\frac{229}{3}$	543	$-\frac{5621}{72}$	$\frac{4550\,057}{23\,100}$	0
221	$-\frac{15}{16}$	$\frac{21}{8}$	0	444	$\frac{77}{8}$	$-\frac{186\,003}{7700}$	0
321	$-\frac{125}{36}$	$\frac{269}{30}$	0	544	$\frac{1155}{32}$	$-\frac{560\,001}{6160}$	0
421	$-\frac{229}{16}$	$\frac{1251}{40}$	1	550	$\frac{197\,045}{108}$	$-\frac{101\,441\,689}{22\,050}$	0
521	$-\frac{937}{12}$	$\frac{27\,059}{350}$	24	551	$-\frac{12\,023}{8}$	$\frac{18\,569\,853}{4900}$	0
222	$\frac{5}{8}$	$-\frac{27}{20}$	0	552	$\frac{1683}{2}$	$-\frac{5718\,309}{2695}$	0
322	$\frac{35}{16}$	$-\frac{213}{40}$	0	553	$-\frac{5159}{16}$	$\frac{2504\,541}{3080}$	0
422	$\frac{35}{3}$	$-\frac{1024}{35}$	0	554	$\frac{24\,563}{312}$	$-\frac{1527\,851}{7700}$	0
522	$\frac{509}{8}$	$-\frac{4209}{28}$	-2	555	$-\frac{9251}{208}$	$\frac{12\,099\,711}{107\,800}$	0

Note that k is equal to the singular modulus k_3, for which the values of the elliptic integrals are explicitly known:

$$K = \boldsymbol{K}(k) = \frac{3^{1/4}\Gamma^3(\frac{1}{3})}{2^{7/3}\pi} \qquad K' = \boldsymbol{K}(k') = \sqrt{3}\boldsymbol{K}(k) \tag{18}$$

$$E = \boldsymbol{E}(k) = \frac{\pi}{4\sqrt{3}\boldsymbol{K}(k)} + \frac{\sqrt{3}+1}{2\sqrt{3}}\boldsymbol{K}(k) \tag{19}$$

$$E' = \boldsymbol{E}(k') = \frac{\pi}{4\boldsymbol{K}(k)} + \frac{\sqrt{3}-1}{2}\boldsymbol{K}(k) \tag{20}$$

Table 2.

$F(3;l,m,n)=\rho_1 f_0+\rho_2/(\pi^2 f_0)+\rho_3$							
lmn	ρ_1	ρ_2	ρ_3	lmn	ρ_1	ρ_2	ρ_3
000	1	0	0	330	145	186	-107
200	$-\frac{1}{3}$	1	0	530	689	2316	$-\frac{2497}{3}$
400	$\frac{25}{9}$	$-\frac{16}{3}$	0	431	$-\frac{2831}{9}$	$-\frac{2101}{3}$	$\frac{898}{3}$
110	1	0	$-\frac{1}{3}$	332	5	120	$-\frac{88}{3}$
310	$-\frac{5}{3}$	5	$-\frac{1}{3}$	532	$\frac{8941}{9}$	$\frac{1598}{3}$	$-\frac{1697}{3}$
510	$\frac{145}{9}$	$-\frac{91}{3}$	$-\frac{1}{3}$	433	$-\frac{1703}{9}$	$-\frac{502}{3}$	$\frac{368}{3}$
211	$-\frac{1}{3}$	-2	$\frac{2}{3}$	440	$\frac{9059}{3}$	4960	$-\frac{7424}{3}$
411	$-\frac{55}{9}$	$\frac{28}{3}$	$\frac{2}{3}$	541	$-\frac{82\,927}{9}$	$-\frac{56\,735}{3}$	8405
220	9	6	$-\frac{16}{3}$	442	$\frac{8221}{9}$	$\frac{9059}{3}$	-1092
420	$\frac{13}{9}$	$\frac{275}{3}$	$-\frac{64}{3}$	543	$-\frac{159\,011}{45}$	$-\frac{12\,800}{3}$	$\frac{7645}{3}$
321	$-\frac{25}{3}$	-29	$\frac{31}{3}$	444	$\frac{19\,039}{15}$	-464	-464
521	$-\frac{779}{9}$	$\frac{95}{3}$	$\frac{95}{3}$	550	$\frac{218\,483}{3}$	133 150	$-\frac{188\,225}{3}$
222	-3	15	-2	552	$\frac{1596\,257}{45}$	$\frac{249\,428}{3}$	$-34\,694$
422	$\frac{737}{9}$	$-\frac{62}{3}$	-32	554	$\frac{111\,001}{9}$	$\frac{20\,534}{3}$	$-\frac{21\,226}{3}$

going back to Legendre; see Whittaker and Watson [9, section 22.81]. Here, $\boldsymbol{E}(k)$ is the complete elliptic integral of the second kind with modulus k, while k' is the complementary modulus given by

$$k' = (1-k^2)^{1/2} = \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

According to Morita [10], $F(3;l,m,n)$ is expressible rationally in terms of f_0, $f_2 = F(3;2,0,0)$, and $F(3;2,2,0)$. Inoue [11, form (3.16), (3.17), (3.18b)] has found that

$$f_2 = \frac{1}{\pi^2}[KK' + 16EE' - 4(2+\sqrt{3})KE' - 4(2-\sqrt{3})K'E] \tag{21}$$

$$F(3;2,2,0) = -f_0 + 2f_2 - \frac{16}{3} + \frac{32}{\pi^2}[KK' + 2EE' - KE' - K'E]. \tag{22}$$

By means of (18)–(20) these expressions are reduced to

$$f_2 = -\frac{1}{3}f_0 + \frac{1}{\pi^2 f_0} \tag{23}$$

$$F(3;2,2,0) = 9f_0 + \frac{6}{\pi^2 f_0} - \frac{16}{3}. \tag{24}$$

Consequently, all $F(3;l,m,n)$ are expressible in the form

$$F(3;l,m,n) = \rho_1 f_0 + \frac{\rho_2}{\pi^2 f_0} + \rho_3 \tag{25}$$

where ρ_1, ρ_2, ρ_3 are rational numbers, as in the simple cubic case. Various values of ρ_1, ρ_2, ρ_3, obtained by means of Morita's [10] recurrence formulae, are shown in table 2.

Finally, we consider the body-centred cubic lattice Green function given by

$$B(z;l,m,n) = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\cos(lu)\cos(mv)\cos(nw)}{z-\cos u\cos v\cos w}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \tag{26}$$

for $z \geqslant 1$, and l, m, n having the same parity ($B = 0$ if $l+m$, $m+n$, or $l+n$ is odd). Watson [1] showed that

$$b_0 = B(1; 0, 0, 0) = \frac{4}{\pi^2}\left[\boldsymbol{K}\left(\frac{1}{\sqrt{2}}\right)\right]^2 = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3} = 1.393\,203\,9297. \tag{27}$$

Although he did not point it out explicitly, it is clear from Joyce's work [12] that $B(1; l, m, n)$ is expressible in the form

$$B(1; l, m, n) = \sigma_1 b_0 + \frac{\sigma_2}{\pi^2 b_0} + \sigma_3 \tag{28}$$

where σ_1, σ_2 and σ_3 are rational numbers. A table of values of σ_1, σ_2 and σ_3, can easily be gathered from the exact expressions for $B(1; l, m, n)$ in [12, appendix A].

3. Discussion

The results in (14), (25), and (28) give rise to the question whether a similar formula holds for other three-dimensional lattice Green functions, such as for the hexagonal lattice. For the cubic lattices, this seems to come about because Watson's expressions involve elliptic integrals of singular moduli and in these cases K', E and E' are all expressible 'rationally' in terms of K (as are their derivatives). It will be interesting to investigate whether this is the case for the other Bravais lattices as well.

As a simple application of our results, we examine the effective resistance $R(l, m, n)$ between the grid points $(0, 0, 0)$ and (l, m, n) in a simple cubic lattice network constructed of unit resistors. According to Cserti [13], $R(l, m, n)$ is expressible as

$$R(l, m, n) = g_0 - G(3; l, m, n). \tag{29}$$

It is a standard textbook exercise to show that $R(1, 0, 0) = \frac{1}{3}$. The next-nearest, and next-next-nearest neighbour resistances are

$$\begin{aligned} R(1, 1, 0) &= \frac{7}{12} g_0 + \frac{1}{2\pi^2 g_0} \\ &= \frac{7(\sqrt{3}-1)}{1152\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) + \frac{24(\sqrt{3}+1)\pi}{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})} = 0.395\,079\,1523 \end{aligned}$$

and

$$\begin{aligned} R(1, 1, 1) &= \frac{9}{8} g_0 - \frac{3}{4\pi^2 g_0} \\ &= \frac{3(\sqrt{3}-1)}{256\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) - \frac{36(\sqrt{3}+1)\pi}{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})} = 0.418\,305\,3109 \end{aligned}$$

respectively, in accordance with [13, table 1]. These, and all other values, are thus rationally expressible in terms of $\Gamma(\frac{1}{24})$, $\Gamma(\frac{11}{24})$, $\sqrt{3}$, and π.

The Green function $G(z; l, m, n)$ can also be represented by the integrals [13]

$$\begin{aligned} G(z; l, m, n) &= \int_0^\infty \mathrm{e}^{-zx} I_l(x) I_m(x) I_n(x)\,\mathrm{d}x \qquad (30) \\ &= \mathrm{i}^{l+m+n+1}\int_0^\infty \mathrm{e}^{-\mathrm{i}zx} J_l(x) J_m(x) J_n(x)\,\mathrm{d}x \qquad (31) \end{aligned}$$

where J_l and I_l are the ordinary and modified Bessel functions of the first kind of order l. Since $G(z; l, m, n)$ is real for $z \geqslant 3$, it follows from (31) that

$$\int_0^\infty \cos(zx) J_l(x) J_m(x) J_n(x)\,\mathrm{d}x = 0 \qquad \text{if} \quad l+m+n \text{ is even}$$
$$\int_0^\infty \sin(zx) J_l(x) J_m(x) J_n(x)\,\mathrm{d}x = 0 \qquad \text{if} \quad l+m+n \text{ is odd.}$$

On setting $z = 3$ in (30) and (31), our previous results furnish the values of several Bessel function integrals, such as

$$\int_0^\infty \mathrm{e}^{-3x} I_0^3(x)\,\mathrm{d}x = \int_0^\infty \sin(3x) J_0^3(x)\,\mathrm{d}x = \frac{\sqrt{3}-1}{96\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right)$$
$$\int_0^\infty \cos(3x) J_0^2(x) J_1(x)\,\mathrm{d}x = \frac{1}{3} - \frac{\sqrt{3}-1}{96\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right)$$
$$\int_0^\infty \sin(3x) J_0(x) J_1^2(x)\,\mathrm{d}x = -\frac{5(\sqrt{3}-1)}{1152\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) + \frac{24(\sqrt{3}+1)\pi}{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})}$$
$$\int_0^\infty \mathrm{e}^{-3x} I_1^3(x)\,\mathrm{d}x = \int_0^\infty \cos(3x) J_1^3(x)\,\mathrm{d}x$$
$$= -\frac{\sqrt{3}-1}{768\pi^3}\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) + \frac{36(\sqrt{3}+1)\pi}{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})}.$$

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